Sage 4.2 Verification

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November 1, 2009

Sage 4.2 was asked to solve the following indefinite integral, which is related to the problem of parabolic arc length.

$$\int \sqrt{1+x^2}\,dx \tag{1}$$

Below I've worked out the pencil-and-paper steps in this integration. By the time I'm through, you will appreciate Computer Algebra Systems like Sage a bit more!

Trigonometric Substitution

Let $x = \sinh u$ and $dx = \cosh u\,du$. (Therefore, in our final solution, $u = \operatorname{arcsinh} x$.) Equation (1) becomes:

$$\int \sqrt{1+\sinh^2 u}\cosh u\,du \tag{2}$$

Using $\sinh^2 u + 1 = \cosh^2 u$:

$$\int \sqrt{\cosh^2 u}\cosh u\,du = \int \cosh^2 u\,du \tag{3}$$

This integral evaluates to our (unfinished) solution

$$\frac{1}{4}\sinh 2u + \frac{u}{2} + C \tag{4}$$

Reversing the Substitution

To recover our solution in terms of x, the first step is to apply the identity $\sinh 2u = 2\sinh u \cosh u$:

$$\frac{1}{4}(2\sinh u \cosh u) + \frac{u}{2} + C \tag{5}$$

We can combine the coefficients $\frac{1}{4}$ and 2. Now, substituting $u = \operatorname{arcsinh} x$:

$$\frac{1}{2}(\sinh \operatorname{arcsinh} x)(\cosh \operatorname{arcsinh} x) + \frac{1}{2}\operatorname{arcsinh} x + C = \tag{6}$$

$$\frac{1}{2}x(\cosh \operatorname{arcsinh} x) + \frac{1}{2}\operatorname{arcsinh} x + C \tag{7}$$

Forward-Reverse Composition

We are almost there. The one sticking point is that "hyperbolic cosine of an inverse hyperbolic sine" term, a functional composition. Fortunately we have the following identity:

$$\operatorname{arcsinh} x = \operatorname{arccosh} \sqrt{1 + x^2} \tag{8}$$

Taking the hyperbolic cosine of both sides:

$$\cosh \operatorname{arcsinh} x = \sqrt{1 + x^2} \tag{9}$$

Solution

Using this final piece of the puzzle, we have our solution. When equation (9) is applied in (7):

$$\frac{1}{2}x\sqrt{1 + x^2} + \frac{1}{2}\operatorname{arcsinh} x + C \tag{10}$$